Applications of Differentiation

Relative and Absolute Extrema

If c is in the domain D of f, then $f(c)$ is the

- absolute maximum if $f(c) \geqslant f(x)$ for all x in D
- absolute minimum if $f(c) \leqslant f(x)$ for all x in D

Also, $f(c)$ is a

- relative maximum if $f(c) \geqslant f(x)$ for x near c
- relative minimum if $f(c) \leqslant f(x)$ for x near c

Extreme Value Theorem

If f is continuous on $[a, b]$, then f attains an absolute minimum and an absolute maximum within $[a, b]$.

Critical Numbers

If c is in the domain of f, then c is a critical number if $f'(c) = 0$ or $f'(c)$ does not exist.

Fermat's Theorem

If f has a relative extremum at c and $f'(c)$ exists, then $f'(c) = 0$.

Closed Interval Method

To find the absolute extrema of a continuous function f on $[a, b]$, use the following steps:

1. Evaluate f at the critical numbers in (a, b).
2. Evaluate f at a and b.
3. The largest value in Steps 1 and 2 is the absolute maximum, while the smallest value is the absolute minimum.

Mean Value Theorem and Rolle's Theorem

If f is continuous on $[a, b]$ and differentiable on (a, b), then a number c exists in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Rolle's Theorem is the special case in which $f'(c) = 0$.

Increasing and Decreasing

- f is increasing on an interval if $f' > 0$ on that interval.
- f is decreasing on an interval if $f' < 0$ on that interval.

Concavity

- The graph of f is concave up on an interval if $f'' > 0$ on that interval.
- The graph of f is concave down on an interval if $f'' < 0$ on that interval.

First-Derivative Test

Let c be a critical number of f.

- f has a relative minimum at c if f' changes sign from negative to positive at c.
- f has a relative maximum at c if f' changes sign from positive to negative at c.
- f has no relative extremum at c if f' does not change sign at c.

Second-Derivative Test

- If $f'(c) = 0$ and $f''(c) > 0$, then f has a relative minimum at c.
- If $f'(c) = 0$ and $f''(c) < 0$, then f has a relative maximum at c.
- If $f'(c) = 0$ but $f''(c) = 0$ or $f''(c)$ does not exist, then the test is inconclusive.

Particle Motion

Suppose that a particle moves in a straight line for time $t \geqslant 0$ with position $s(t)$, velocity $v(t)$, and acceleration $a(t)$.

$$v(t) = \frac{\mathrm{d}s}{\mathrm{d}t} \qquad a(t) = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2}$$

- The particle's displacement on $[t_1, t_2]$ is $\Delta s = s(t_2) - s(t_1)$.
- The particle's speed is $|v(t)|$.
- The particle is speeding up if v and a have the same sign, and slowing down if v and a have opposite signs.

L'Hôpital's Rule

If f and g are differentiable with $g'(x) \neq 0$ near a, and $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ both equal 0 or $\pm\infty$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right exists.

Applications of Differentiation

Indeterminate Limit Form $0 \times \infty$

If $\lim_{x\to a} p(x) = 0$ and $\lim_{x\to a} q(x) = \pm\infty$, then rewrite the indeterminate limit $\lim_{x\to a} [p(x)\, q(x)]$ as

$$\lim_{x\to a} \frac{p(x)}{1/q(x)} \qquad \text{or} \qquad \lim_{x\to a} \frac{q(x)}{1/p(x)}$$

and use L'Hôpital's Rule.

Indeterminate Limit Form $\infty - \infty$

If $\lim_{x\to a} p(x) = \infty$ and $\lim_{x\to a} q(x) = \infty$, then rewrite the indeterminate limit $\lim_{x\to a} [p(x) - q(x)]$ by factoring, rationalizing, or combining fractions, and apply L'Hôpital's Rule as necessary.

Indeterminate Limit Forms 0^0, 1^∞, and ∞^0

If $\lim_{x\to a} [p(x)]^{q(x)}$ is in the indeterminate form 0^0, 1^∞, or ∞^0, then use the following steps:

1. Let $y = [p(x)]^{q(x)}$; then $\ln y = q(x) \ln [p(x)]$.
2. Evaluate $L = \lim_{x\to a} \ln y$.
3. Conclude that $\lim_{x\to a} y = e^L$.

Relative Rates of Growth

Consider the limit $L = \lim_{x\to\infty} \dfrac{f(x)}{g(x)}$.

- If $L = \infty$, then f grows faster than g.
- If $L = 0$, then g grows faster than f.
- If L is a positive constant, then f and g grow at the same rate.

Optimization Problems

Use the following steps:

1. Understand the situation by drawing diagrams and testing various possibilities.
2. Assign a variable—say, Q—to the quantity to maximize or minimize. Introduce notation by assigning algebraic quantities to unknown values. It is helpful to use suggestive variables (e.g., A for area, d for distance, L for length).
3. Express Q in terms of one variable—say, x—to obtain $Q(x)$.
4. Use calculus to find the absolute minimum or maximum of $Q(x)$. Find the critical numbers of Q and consider the endpoints of the domain in context.

Business Functions

If a firm produces x units and $p(x)$ is a demand function, then the revenue function is

$$R(x) = x\, p(x)$$

If $C(x)$ is the cost function, then the profit function is

$$P(x) = R(x) - C(x)$$

The average cost function, for $x > 0$, is

$$\overline{C}(x) = \frac{C(x)}{x}$$

Marginal Functions

- Marginal cost function: $C'(x)$
- Marginal revenue function: $R'(x)$
- Marginal profit function: $P'(x)$

Newton's Method

An approximation of a root r of a function f is given by the iterative formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$